Detailed derivation of analytical equations used for response spectrum generation

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1 Preliminaries

Equation of motion for a single degree of freedom can be written as

$$m\ddot{u} + c\dot{u} + ku = -m\ddot{u}_g(t) \tag{1}$$

where u is displacement relative to the ground [3], [4].
Eqn. (1) can be written as modal equation

$$\ddot{u} + 2\xi\omega\dot{u} + \omega^2 u = -\ddot{u}_g(t) \tag{2}$$

where

$$\omega = \sqrt{\frac{k}{m}} \tag{3}$$

and

$$\xi = \frac{c}{2\sqrt{km}} < 1 \tag{4}$$

Ground acceleration $\ddot{u}_g$ is assumed to be linear over a time step i:

$$\ddot{u}_g(t) = \ddot{u}_g(t_{i-1}) + \frac{\ddot{u}_g(t_i) - \ddot{u}_g(t_{i-1})}{t_i - t_{i-1}}(t - t_{i-1}) \tag{5}$$

Let's define slope, s, of the acceleration within a time step as

$$s = \frac{\ddot{u}_g(t_i) - \ddot{u}_g(t_{i-1})}{t_i - t_{i-1}} \tag{6}$$

and time step independent time variable, $\bar{t}$, as

$$\bar{t} = t - t_{i-1} \tag{7}$$

For each time step, Equations (2), (5), (6) and (7) can be combined into the following differential equation for $u(\bar{t})$

$$\ddot{u}(\bar{t}) + 2\xi\omega\dot{u}(\bar{t}) + \omega^2 u(\bar{t}) = -\ddot{u}_g(t_{i-1}) - s\bar{t} \tag{8}$$

with initial conditions

$$u(\bar{t})_{t=0} = u(t_{i-1}) \tag{9}$$

$$\dot{u}(\bar{t})_{t=0} = \dot{u}(t_{i-1}) \tag{10}$$

The solution of Equation (8) with initial conditions (9) and (10) will be obtained as a sum of homogeneous and particular solution

$$u = u_h + u_p \tag{11}$$

2 Homogeneous Solution

2.1 General Solution

The solution of homogeneous second-order linear ordinary differential equation with constant coefficients

$$y'' + ay' + by = 0 \tag{12}$$

where a, b are real constants is readily available in the literature [1]. Let $\lambda_1$and λ_2 be the roots of the characteristic equation

$$f(\lambda) = \lambda^2 + a\lambda + b = 0 \tag{13}$$

1. If $\lambda_1 \neq \lambda_2$ real, then

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_1 x} \tag{14}$$

2. If $\lambda = \lambda_1 = \lambda_2$, then

$$y = (C_1 x + C_2)e^{\lambda x} \tag{15}$$

3. If $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$ ($\beta \neq 0$), then

$$y = e^{\alpha x}(C_1 \cos(\beta x) + C_2 \sin(\beta x)) = e^{\alpha x} C \cos(\beta x + \theta) \tag{16}$$

2.2 Problem Specific Solution

By comparison of Eqns. (1) and (12), the generic a, b coefficients in Eqn. (12) can we written in the problem specific variables ξ and ω as

$$a = 2\xi\omega \quad (17)$$
$$b = \omega^2 \quad (18)$$

Then

$$a^2 - 4b = (2\xi\omega)^2 - 4\omega^2 = 4\omega^2(\xi^2 - 1) = -4\omega^2(1 - \xi^2) = -4{\omega_d}^2 < 0 \quad (19)$$

where

$$\omega_d = \omega\sqrt{1 - \xi^2} \quad (20)$$

The solution of characteristic equation (13) has complex roots

$$\lambda_1 = \frac{-2\xi\omega + 2i\omega\sqrt{1 - \xi^2}}{2} = -\xi\omega + i\omega_d \quad (21)$$
$$\lambda_2 = -\xi\omega - i\omega_d \quad (22)$$

α and β from the general solution can be rewritten using the problem specific variables as

$$\alpha = -\xi\omega \quad (23)$$
$$\beta = \omega_d \quad (24)$$

Substituting Eqns. (23) and (24) into Eqn. (16) yields

$$u_h(\bar{t}) = e^{-\xi\omega\bar{t}}\left(C_1 \cos(\omega_d\bar{t}) + C_2 \sin(\omega_d\bar{t})\right) \quad (25)$$

The constants C_1 and C_2 will be determined from the initial conditions.

3 Particular Solution

3.1 General Solution

Particular solution, y_p, of differential equation

$$y'' + ay' + by = R(x) \quad (26)$$

where a, b are real constants is also readily available in the literature [1]. For the special case of $R(x) = P(x) = Ax^2 + Bx + C$ and $b \neq 0$

$$y_p = \frac{1}{b}\left(P(x) - \frac{a}{b}P'(x) + \frac{a^2 - b}{b^2}P''(x)\right) \tag{27}$$

For our case, $A = 0$, therefore

$$\begin{aligned} P(x) &= Bx + C & (28)\\ P'(x) &= B & (29)\\ P''(x) &= 0 & (30) \end{aligned}$$

and Eqn. (27) reduces to

$$y_p = \frac{1}{b}\left(Bx + C - \frac{a}{b}B\right) \tag{31}$$

3.2 Problem Specific Solution

The a, b, B, and C constants can be written using problem specific notation (see Eqns. (8), (17) and (18)) as

$$\begin{aligned} a &= 2\xi\omega & (32)\\ b &= \omega^2 & (33)\\ B &= -s & (34)\\ C &= -\ddot{u}_g(t_{i-1}) & (35) \end{aligned}$$

Then the general particular solution described by Eqn. (27) becomes

$$\begin{aligned} u_p(\bar{t}) &= \frac{1}{\omega^2}\left(-s\bar{t} + (-\ddot{u}(t_{i-1})) - \frac{2\xi\omega}{\omega^2}(-s)\right)\\ &= -\frac{s}{\omega^2}\bar{t} + \frac{1}{\omega^2}\left(\frac{2\xi}{\omega}s - \ddot{u}_g(t_{i-1})\right)\\ &= E\bar{t} + F & (36) \end{aligned}$$

where

$$\begin{aligned} E &= -\frac{s}{\omega^2} & (37)\\ F &= \frac{1}{\omega^2}\left(\frac{2\xi}{\omega}s - \ddot{u}_g(t_{i-1})\right) & (38) \end{aligned}$$

4 Complete Solution

Eqns. (25) and (36) can be added to obtain complete solution as

$$\begin{aligned} u(\bar{t}) &= u_h(\bar{t}) + u_p(\bar{t}) \\ &= e^{-\xi\omega\bar{t}}\left(C_1\cos(\omega_d\bar{t}) + C_2\sin(\omega_d\bar{t})\right) + E\bar{t} + F \end{aligned} \tag{39}$$

The unknown coefficients C_1 and C_2 will be determined from initial conditions.

Derivative of Eqn. (39) is

$$\begin{aligned} \dot{u}(\bar{t}) &= -\xi\omega e^{-\xi\omega\bar{t}}\left(C_1\cos(\omega_d\bar{t}) + C_2\sin(\omega_d\bar{t})\right) + \\ & \quad e^{-\xi\omega\bar{t}}\left(C_1\omega_d(-\sin(\omega_d\bar{t})) + C_2\omega_d\cos(\omega_d\bar{t})\right) + E \end{aligned} \tag{40}$$

Evaluating Eqn. (39) and (40) for initial conditions (9) and (10) yields

$$u(t_{i-1}) = e^0(C_1\cos 0 + C_2\sin 0) + F \tag{41}$$

$$\begin{aligned} \dot{u}(t_{i-1}) &= -\xi\omega e^0(C_1\cos 0 + C_2\sin 0) + \\ & \quad e^0(C_1\omega_d(-\sin 0) + C_2\omega_d\cos 0) + E \end{aligned} \tag{42}$$

Simplify the above as

$$u(t_{i-1}) = C_1 + F \tag{43}$$

$$\dot{u}(t_{i-1}) = -\xi\omega C_1 + C_2\omega_d + E \tag{44}$$

Finally, express C_1 and C_2

$$C_1 = u(t_{i-1}) - F \tag{45}$$

$$C_2 = \frac{\dot{u}(t_{i-1}) + \xi\omega C_1 - E}{\omega_d} \tag{46}$$

4.1 Summary

The final solution is

$$u(\bar{t}) = e^{-\xi\omega\bar{t}}\left(C_1\cos(\omega_d\bar{t}) + C_2\sin(\omega_d\bar{t})\right) + E\bar{t} + F \tag{47}$$

where

$$
\begin{aligned}
C_1 &= u(t_{i-1}) - F && (48)\\
C_2 &= \frac{\dot{u}(t_{i-1}) + \xi\omega C_1 - E}{\omega_d} && (49)\\
E &= -\frac{s}{\omega^2} && (50)\\
F &= \frac{1}{\omega^2}\left(\frac{2\xi}{\omega}s - \ddot{u}_g(t_{i-1})\right) && (51)\\
s &= \frac{\ddot{u}_g(t_i) - \ddot{u}_g(t_{i-1})}{t_i - t_{i-1}} && (52)\\
\omega_d &= \omega\sqrt{1-\xi^2} && (53)\\
\bar{t} &= t - t_{i-1} && (54)
\end{aligned}
$$

5 Computer Algorithm

The formulation described by Eqn. (39) is applied recursively for each step, $i = 1$ to n. In order to apply the algorithm and to obtain acceleration response spectrum, we will also need to calculate velocity and acceleration by obtaining first and second derivatives of Eqn. (39).

5.1 Velocity

$$
\begin{aligned}
\dot{u}(\bar{t}) &= -\xi\omega e^{-\xi\omega\bar{t}}\left(C_1\cos(\omega_d\bar{t}) + C_2\sin(\omega_d\bar{t})\right) + \\
&\quad e^{-\xi\omega\bar{t}}\left(C_1\omega_d(-\sin(\omega_d\bar{t})) + C_2\omega_d\cos(\omega_d\bar{t})\right) + E && (55)
\end{aligned}
$$

5.2 Acceleration

$$
\begin{aligned}
\ddot{u}(\bar{t}) &= \xi^2\omega^2 e^{-\xi\omega\bar{t}}\left(C_1\cos(\omega_d\bar{t}) + C_2\sin(\omega_d\bar{t})\right) - \\
&\quad \xi\omega e^{-\xi\omega\bar{t}}\left(C_1\omega_d(-\sin(\omega_d\bar{t})) + C_2\omega_d\cos(\omega_d\bar{t})\right) - \\
&\quad \xi\omega e^{-\xi\omega\bar{t}}\left(C_1\omega_d(-\sin(\omega_d\bar{t})) + C_2\omega_d\cos(\omega_d\bar{t})\right) + \\
&\quad e^{-\xi\omega\bar{t}}\left(C_1{\omega_d}^2(-\cos(\omega_d\bar{t})) + C_2{\omega_d}^2(-\sin(\omega_d\bar{t}))\right) \\
&= \xi^2\omega^2 e^{-\xi\omega\bar{t}}\left(C_1\cos(\omega_d\bar{t}) + C_2\sin(\omega_d\bar{t})\right) - \\
&\quad 2\xi\omega e^{-\xi\omega\bar{t}}\left(C_1\omega_d(-\sin(\omega_d\bar{t})) + C_2\omega_d\cos(\omega_d\bar{t})\right) + \\
&\quad e^{-\xi\omega\bar{t}}\left(C_1{\omega_d}^2(-\cos(\omega_d\bar{t})) + C_2{\omega_d}^2(-\sin(\omega_d\bar{t}))\right) && (56)
\end{aligned}
$$

5.3 Algorithm

The ground acceleration time history is given as a vector of acceleration values that are uniformly spaced in time as shown in Fig. 1.

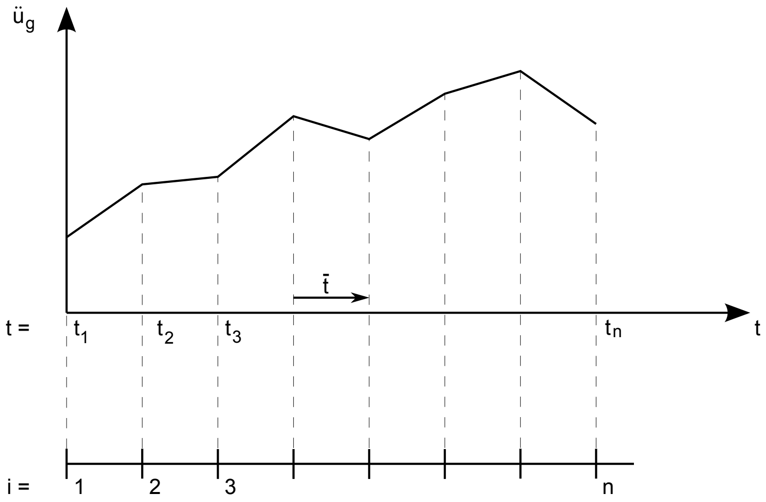

Figure 1: Measured ground acceleration discretized by linear segments

In the first solution step, the initial displacement and velocity at time $t_1 = 0$ are taken to be zero. Eqns. (47), (55) and (56) are applied to obtain displacement, velocity, and acceleration the time t_2.

In the subsequent steps, the displacement and velocity calculated for the previous step are applied as initial conditions for the current step and displacement, velocity, and acceleration at the end of the step are calculated.

After repeating the above procedure for all n steps, time histories of displacements, velocities and accelerations are obtained. Spectral values are determined as follows [2]:

Spectral displacement, $\mathbf{S_d}$ Maximum absolute displacement obtained from the time history.

Spectral velocity, $\mathbf{S_v}$ Maximum absolute velocity obtained from the time history.

Spectral acceleration, $\mathbf{S_a}$ Maximum absolute acceleration obtained from the time history.

Spectral pseudo-velocity, $\mathbf{S_{pv}}$ Calculated from spectral displacement as $S_{pv} = \omega S_d$

Spectral pseudo-acceleration, $\mathbf{S_{pa}}$ Calculated from spectral displacement as $S_{pa} = \omega^2 S_d$

References

[1] F. Bubeník, M. Pultar, and I. Pultarová. *Matematické vzorce a metody.* Vydavatelství ČVUT, second edition, 1997. p. 224 (in Czech).

[2] Anil K. Chopra. *Dynamics of Structures, Theory and Applications to Earthquake Engineering.* Prentice Hall of India, New Delhi, 2nd edition, 2002. section 6-6, p. 208.

[3] Ray W. Clough and Joseph Penzien. *Dynamics of Structures.* McGraw-Hill, New York, 2nd edition, 1993. 738 pages, ISBN 0-07-011394-7, section 2-6, p. 26.

[4] Structural Wiki. Single degree of freedom damped free vibrations. Available at http://www.structuralwiki.org/en/ Single_degree_of_freedom_damped_free_vibrations. accessed October 2009.