Detailed derivation of selected link relationships and equations from CSI Analysis Reference Manual (Draft)

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1 Detailed derivation of relationship between joint displacements and link internal forces

This article contains a detailed derivation of some link equations presented in CSI Analysis Reference Manual, chapter "The Link/Support Element–Basic". Uncoupled linear force-deformation relationship can be written using the following matrix equations (see Manual section Link/Support Properties, Uncoupled Linear Force-Deformation Relationships):

$$\begin{Bmatrix} f_{u1} \\ f_{u2} \\ f_{u3} \\ f_{r1} \\ f_{r2} \\ f_{r3} \end{Bmatrix} = \begin{bmatrix} k_{u1} & 0 & 0 & 0 & 0 & 0 \\ & k_{u2} & 0 & 0 & 0 & 0 \\ & & k_{u3} & 0 & 0 & 0 \\ & & & k_{r1} & 0 & 0 \\ & sym. & & & k_{r2} & 0 \\ & & & & & k_{r3} \end{bmatrix} \begin{Bmatrix} d_{u1} \\ d_{u2} \\ d_{u3} \\ d_{r1} \\ d_{r2} \\ d_{r3} \end{Bmatrix} \tag{1}$$

The above equation can be concisely written as:

$$\{f_u\} = [K]\{d_u\} \tag{2}$$

Link element forces can be written in the terms of link internal forces as follows (see Manual section Link/Support Properties, Element Internal Forces):

$$\begin{Bmatrix} P \\ V2 \\ V3 \\ T \\ M2 \\ M3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & d - \mathbf{dj3} & 0 & 1 & 0 \\ 0 & d - \mathbf{dj2} & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} f_{u1} \\ f_{u2} \\ f_{u3} \\ f_{r1} \\ f_{r2} \\ f_{r3} \end{Bmatrix} \tag{3}$$

Concisely:

$$\{F\} = [M_1]\{f_u\} \tag{4}$$

For one-joint link, link deformations can be written in the terms of joint displacements as follows (note that d = 0 at J end of the link; see Manual section Internal Deformations):

$$\begin{Bmatrix} d_{u1} \\ d_{u2} \\ d_{u3} \\ d_{r1} \\ d_{r2} \\ d_{r3} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -\mathbf{dj2} \\ 0 & 0 & 1 & 0 & \mathbf{dj3} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ r_1 \\ r_2 \\ r_3 \end{Bmatrix}_j \tag{5}$$

Concisely:

$$\{d_u\} = [M_2]\{u\} \tag{6}$$

Equations (4) and (6) can be substituted into equation (2) to yield:

$$[M_1]^{-1}\{F\} = [K][M_2]\{u\} \tag{7}$$

$$\{F\} = [M_1][K][M_2]\{u\} \tag{8}$$

Upon substitution:

$$[M_1][K] = \begin{bmatrix} k_{u1} & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{u2} & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{u3} & 0 & 0 & 0 \\ 0 & 0 & 0 & k_{r1} & 0 & 0 \\ 0 & 0 & -\mathbf{dj3}k_{u3} & 0 & k_{r2} & 0 \\ 0 & -\mathbf{dj2}k_{u2} & 0 & 0 & 0 & k_{r3} \end{bmatrix} \tag{9}$$

$$[M_1][K][M_2] = \begin{bmatrix} k_{u1} & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{u2} & 0 & 0 & 0 & -\mathbf{dj2}k_{u2} \\ 0 & 0 & k_{u3} & 0 & \mathbf{dj3}k_{u3} & 0 \\ 0 & 0 & 0 & k_{r1} & 0 & 0 \\ 0 & 0 & -\mathbf{dj3}k_{u3} & 0 & -k_{r2} - \mathbf{dj3}^2 k_{u3} & 0 \\ 0 & -\mathbf{dj2}k_{u2} & 0 & 0 & 0 & k_{r3} + \mathbf{dj2}^2 k_{u2} \end{bmatrix} \tag{10}$$

Or:

$$\left\{ \begin{array}{c} P \\ V2 \\ V3 \\ T \\ M2 \\ M3 \end{array} \right\}_j = \left[\begin{array}{cccccc} k_{u1} & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{u2} & 0 & 0 & 0 & -\mathbf{dj2}k_{u2} \\ 0 & 0 & k_{u3} & 0 & \mathbf{dj3}k_{u3} & 0 \\ 0 & 0 & 0 & k_{r1} & 0 & 0 \\ 0 & 0 & -\mathbf{dj3}k_{u3} & 0 & -k_{r2} - \mathbf{dj3}^2 k_{u3} & 0 \\ 0 & -\mathbf{dj2}k_{u2} & 0 & 0 & 0 & k_{r3} + \mathbf{dj2}^2 k_{u2} \end{array} \right] \left\{ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ r_1 \\ r_2 \\ r_3 \end{array} \right\}_j \tag{11}$$

2 Equivalent Shear and Bending Springs for Prismatic Beam

The Manual gives an example of equivalent shear and bending springs computed for prismatic beam with a section bending stiffness of EI in the 1-2 plane. The beam element stiffness matrix at joint **j** for the 1-2 bending plane is:

$$\left\{ \begin{array}{c} V2 \\ M3 \end{array} \right\}_j = \frac{EI}{L^3} \left[\begin{array}{cc} 12 & -6L \\ -6L & 4L^2 \end{array} \right] \left\{ \begin{array}{c} u_2 \\ r_3 \end{array} \right\}_j \tag{12}$$

From Equation (11), the following two equations can be extracted:

$$\left\{ \begin{array}{c} V2 \\ M3 \end{array} \right\}_j = \left[\begin{array}{cc} k_{u2} & -\mathbf{dj2}k_{u2} \\ -\mathbf{dj2}k_{u2} & k_{r3} + \mathbf{dj2}^2 k_{u2} \end{array} \right] \left\{ \begin{array}{c} u_2 \\ r_3 \end{array} \right\}_j \tag{13}$$

Equating the matrix coefficients in Equations (12) and (13) yields the following set of three equations for three unknowns k_{u2}, k_{r3} and **dj2**:

$$k_{u2} = \frac{12EI}{L^3} \tag{14}$$

$$-\mathbf{dj2}k_{u2} = \frac{-6EI}{L^2} \tag{15}$$

$$k_{r3} + \mathbf{dj2}^2 k_{u2} = \frac{4EI}{L} \tag{16}$$

From Equation (15):

$$\mathbf{dj2} = \frac{-6EI}{L^2}\frac{1}{-k_{u2}} = \frac{-6EI}{L^2}\frac{-12EI}{L^3} = \frac{L}{2} \tag{17}$$

From Equation (16):

$$k_{r3} = \frac{4EI}{L} - \mathbf{dj2}^2 k_{u2} = \frac{4EI}{L} - \left(\frac{L}{2}\right)^2 \frac{12EI}{L^3} = \frac{EI}{L} \tag{18}$$